

Exercises for 'Topics in complex analysis'

(15/10/2025)

H 6.1 (Examples of infinite products of functions)

Examine for which $z \in \mathbb{C}$ the following products converge absolutely and determine the largest open set $U \subset \mathbb{C}$ on which they converge locally normally:

$$\text{a) } \prod_{n=1}^{\infty} (1 + z^n) \quad \text{b) } \prod_{n=1}^{\infty} \left(1 + \frac{z^n}{n!}\right) \quad \text{c) } \prod_{n=1}^{\infty} \cos(2^{-n}z)$$

H 6.2 (Further practice on infinite products)

Show that the product

$$H(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges locally normally on \mathbb{C} .

Remark: One can show that the limit $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log(n)\right)$ exists. Then $\Gamma(z) := e^{-\gamma z} \frac{1}{H(z)}$ yields an alternative representation of the Γ function.

H 6.3 (The product formula for the sine)

The goal of this exercise is to derive the product formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \forall z \in \mathbb{C}.$$

a) Use the partial fraction decomposition $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$ (cf. Exercise H 4.2) to show

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad \forall z \in \mathbb{C} \setminus \mathbb{Z},$$

with a suitable notion of convergence for the series on the right hand side.

Hint: You may use without proof the following fact from analysis: if $D \subset \mathbb{C}$ is a domain and $f_n, f : D \rightarrow \mathbb{C}$ are continuously differentiable functions such that $f_n(z_0) \rightarrow f(z_0)$ for some $z_0 \in D$ and $f'_n \rightarrow f'$ locally uniformly, then $f_n \rightarrow f$ locally uniformly.

b) Define the functions $f_n : \mathbb{C} \rightarrow \mathbb{C}$ by $f_n(z) = 1 - \frac{z^2}{n^2}$ and compute the logarithmic derivative f'/f for $f(z) = \pi z \prod_{n=1}^{\infty} f_n(z)$.

c) Conclude by comparing suitable terms.

H 6.4 (Consequences of the sine product formula)

Use the product formula from Exercise H 6.3 to show the following statements:

$$\text{a) } \frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} \quad \text{b) } \cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right) \quad \text{c)* } \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hint: For c) use a Taylor expansion.